Lecture-13

Introduction of successive differentiation
nth derivative of some elementary functions

Successive differentiation means to differentiate the function successively i.e. to find $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$ where y is the function of x.

Some standard results :-

① If $y = (ax+b)^m$ then

$$y_n = \frac{\lfloor m}{\lfloor (m-n)} a^n (ax+b)^{m-n} \qquad m > n$$

(m is +ve integer)

② If $y = \frac{1}{ax+b}$ then

nth derivative $y_n = \frac{(-1)^n \lfloor n \, a^n}{(ax+b)^{n+1}}$

③ If $y = \log(ax+b)$ then $y_n = \frac{(-1)^{n-1} a^n \lfloor n-1}{(ax+b)^n}$

④ If $y = a^{mx}$ then $y_n = m^n a^{mx} (\log_e a)^n$

⑤ If $y = e^{mx}$ then $y_n = m^n e^{mx}$

⑥ If $y = \sin(ax+b)$ then $y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

⑦ If $y = \cos(ax+b)$ then $y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

⑧ If $y = e^{ax} \sin(bx+c)$ then $y_n = e^{ax}(a^2+b^2)^{n/2} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

⑨ If $y = e^{ax} \cos(bx+c)$ then $y_n = e^{ax}(a^2+b^2)^{n/2} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Imp
Example:- If $y = \frac{1}{1-5x+6x^2}$ find y_n.

Solution:- $y = \frac{1}{1-5x+6x^2} = \frac{1}{(2x-1)(3x-1)}$

$$y = \frac{1}{(2x-1)(3x-1)} = \frac{2}{2x-1} - \frac{3}{3x-1}$$

Differentiating y n times, we get

$$y_n = 2\left[\frac{(-1)^n 2^n \lfloor n}{(2x-1)^{n+1}}\right] - 3\left[\frac{(-1)^n 3^n \lfloor n}{(3x-1)^{n+1}}\right]$$

$$y_n = (-1)^n \lfloor n \left[\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{3^{n+1}}{(3x-1)^{n+1}}\right]$$

Example :- If $y = \log(ax + x^2)$ then find y_n.

Solution :- $y = \log x(x+a)$

$$y = \log x + \log(x+a)$$

Differentiation n times of y gives

$$y_n = \frac{(-1)^{n-1}\lfloor n-1}{x^n} + \frac{(-1)^{n-1}\lfloor n-1}{(x+a)^n}$$

$$y_n = (-1)^{n-1}\lfloor n-1 \left[\frac{1}{x^n} + \frac{1}{(x+a)^n}\right]$$

Example :- If $y = \frac{x^n - 1}{x-1}$ then find y_n.

[2011-12]

Solution :- $y = \frac{x^n - 1}{x-1}$

$$y = x^{n-1} + x^{n-2} + \cdots + x + 1$$

$$y_n = 0$$

Example :- If $y = \sin nx + \cos nx$.

Prove that

$$y_r = n^r\left[1 + (-1)^r \sin 2nx\right]^{1/2}$$

[2011, 2017]

Solution :-

$$y = \sin nx + \cos nx$$

$$y_r = \left[\sin\left[nx + \frac{r\pi}{2}\right] + \cos\left[nx + \frac{r\pi}{2}\right]\right]$$

$$y_r = \left[\left\{\sin\left(nx + \frac{r\pi}{2}\right) + \cos\left(nx + \frac{r\pi}{2}\right)\right\}^2\right.$$

$$y_r = n^r\left[\sin^2\left(nx + \frac{r\pi}{2}\right) + \cos^2\left(nx + \frac{r\pi}{2}\right) + 2\sin\left(\right.\right.$$

$$\cos\left(nx + \frac{r\pi}{2}\right.$$

$$y_r = n^r\left[1 + \sin(2nx + r\pi)\right]^{1/2}$$

$$y_r = n^r\left[1 + \sin 2nx \cos r\pi + \cos 2nx \sin\right.$$

$$y_r = n^r\left[1 + (-1)^r \sin 2nx\right]^{1/2}$$

Example :- Find the nth derivative of $\sin^2 x \cos^3 x$.

Solution :- $y = \sin^2 x \cos^3 x$

$$y = \sin^2 x \cos^2 x \cos x = \frac{1}{4}\sin^2 2x \cos x$$

$$= \frac{1}{4}\times\frac{1}{2}(1-\cos 4x)\cos x \qquad \because \sin^2 2x = \frac{1}{2}(1-\cos 4x)$$

$$= \frac{1}{8}\left(\cos x - \frac{2\cos 4x \cos x}{2}\right)$$

$$= \frac{1}{8}\cos x - \frac{1}{16}\left(\cos 5x + \cos 3x\right)$$

$$y = \frac{1}{8}\cos x - \frac{1}{16}\cos 5x - \frac{1}{16}\cos 3x$$

then

$$y_n = \frac{1}{8}\cos\left(x+\frac{n\pi}{2}\right) - \frac{5^n}{16}\cos\left(5x+\frac{n\pi}{2}\right) - \frac{3^n}{16}\cos\left(3x+\frac{n\pi}{2}\right)$$

Example :- Find the nth derivative of $\sin^4 x$.

Solution:- As above

ans: $$y_n = -\frac{1}{2}\cdot 2^n \cos\left(2x+\frac{n\pi}{2}\right) + \frac{4^n}{8}\cos\left(4x+\frac{n\pi}{2}\right)$$

Lecture 14

Leibnitz's theorem & nth derivative of product of functions

Leibnitz's theorem

Statement :- If u and v are two functions of x such that their n^{th} derivatives exist, then nth derivative of their product is given by

$$D^n(uv) = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_n u\, v_n$$

here $u_n = D^n(u) = \dfrac{d^n u}{dx^n}$

$v_n = D^n(v) = \dfrac{d^n v}{dx^n}$

Example :- Find the nth derivative of $x^{n-1}\log x$.

Solution :- $y = x^{n-1}\log x$ (base e)

$$y_1 = (n-1)x^{n-2}\log x + x^{n-2}$$

Lecture 15

Relation between y_n, y_{n+1} and y_{n+2}

Leibnitz theorem is used to establish such relations

Leibnitz theorem

$$D^n(u\cdot v) = v\,D^n(u) + {}^nc_1\,D^{n-1}(u)\cdot D(v) + {}^nc_2\,D^{n-2}(u)\cdot D^2(v) + \cdots + u\cdot D^n(v)$$

or

$$D^n(u\cdot v) = v u_n + {}^nc_1\, u_{n-1} v_1 + {}^nc_2\, u_{n-2} v_2 + \cdots + u v_n$$

V. Imp

Example :- If $y = e^{m\cos^{-1}x}$ then find the relation between y_n, y_{n+1} and y_{n+2}. [2015, 19]

Solution :- $y = e^{m\cos^{-1}x}$ ... (1)

Differentiating (1) w.r.to x

$$y_1 = e^{m\cos^{-1}x}\cdot\left(\frac{-m}{\sqrt{1-x^2}}\right)$$

$$\sqrt{1-x^2}\cdot y_1 = -my$$

Squaring both sides, we get

$$(1-x^2)y_1^2 = m^2y^2 \quad \cdots (2)$$

Differentiating (2) w.r. to x

$$(-2x)y_1^2 + 2(1-x^2)y_1y_2 = 2m^2yy_1$$

which gives

$$(1-x^2)y_2 - xy_1 - m^2y = 0 \quad \cdots (3)$$

Now differentiating both sides n times w.r. to x by Leibnitz theorem

$$D^n[(1-x^2)y_2] - D^n[xy_1] - m^2D^ny = 0$$

$$(1-x^2)y_{n+2} + {}^nc_1\, y_{n+1}(-2x) + {}^nc_2\, y_n(-2) - xy_{n+1} - {}^nc_1\, y_n - m^2y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - \frac{2n(n-1)}{2}y_n - xy_{n+1} - ny_n - m^2y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n + ny_n - ny_n - m^2y_n = 0$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

Example :- If $y = e^{\tan^{-1}x}$, prove that

$$(1+x^2)y_{n+2} + [(2n+2)x-1]y_{n+1} + n(n+1)y_n = 0$$

[2013, 2017, 2020]

Solution :- same as above

Example :- If $y^{1/m} + y^{-1/m} = 2x$, prove that

$$(x^2-1)\, y_{n+2} + (2n+1)x\, y_{n+1} + (n^2-m^2)\, y_n = 0$$

Solution :- we have $y^{1/m} + y^{-1/m} = 2x$

$$y^{1/m} + \frac{1}{y^{1/m}} = 2x \qquad \cdots (1)$$

Suppose $y^{1/m} = t$, then (1) becomes

$$t^2 + 1 = 2xt \Rightarrow t^2 - 2xt + 1 = 0$$

$$t = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

so, $y^{1/m} = x \pm \sqrt{x^2-1}$

$$y = \left[x \pm \sqrt{x^2-1}\right]^m$$

$$y_1 = m\left[x \pm \sqrt{x^2-1}\right]^{m-1} \cdot \left[1 \pm \frac{1}{2\sqrt{x^2-1}} \cdot 2x\right]$$

$$= m\left[x \pm \sqrt{x^2-1}\right]^{m-1} \left[\frac{x \pm \sqrt{x^2-1}}{\sqrt{x^2-1}}\right]$$

$$\sqrt{x^2-1}\; y_1 = m\left[x \pm \sqrt{x^2-1}\right]^m$$

$$\sqrt{x^2-1}\; y_1 = m y$$

$$(x^2-1)\, y_1^2 = m^2 y^2 \qquad \cdots (2)$$

Again differentiating (2) w.r. to x, we get

$$2x\, y_1^2 + (x^2-1)\, 2y_1 y_2 = 2m^2 y y_1$$

$$\Rightarrow (x^2-1)\, y_2 + x y_1 = m^2 y$$

or

$$(x^2-1)\, y_2 + x y_1 - m^2 y = 0 \qquad \cdots (3)$$

Now differentiating (3) n-times by Leibnitz theorem

$$D^n\left[(x^2-1)y_2\right] + D^n\left[x y_1\right] - m^2 D^n(y) = 0$$

$$(x^2-1)\, y_{n+2} + n y_{n+1} \cdot 2x + \frac{n(n-1)}{2} y_n \cdot 2 + x y_{n+1} + n y_n - m^2 y_n = 0$$

$$(x^2-1)\, y_{n+2} + (2n+1)x\, y_{n+1} + (n^2-m^2)\, y_n = 0$$

Example :- If $y = \left(\frac{1+x}{1-x}\right)^{1/2}$, prove that

$$(1-x^2)\, y_n - \left[2(n-1)x + 1\right] y_{n-1} - (n-1)(n-2)\, y_{n-2} = 0$$

[2011]

Solution :-

Given function

$$y = \left(\frac{1+x}{1-x}\right)^{1/2}$$

Taking logrithm on both sides with base e.

$$\ln y = \frac{1}{2}\left[\ln(1+x) + \ln(1-x)\right] \quad \cdots(1)$$

Differentiating (1) w.r. to x, we get

$$\frac{1}{y}\cdot y_1 = \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right] = \frac{1}{2}\left[\frac{1-x+1+x}{(1+x)(1-x)}\right]$$

$$(1-x^2)\,y_1 = y \quad \cdots(2)$$

Diff. (2) $(n-1)$ times w.r. to x. by Leibnitz theorem

$$(1-x^2)\,y_n + (n-1)\,y_{n-1}(-2x) + \frac{(n-1)(n-2)}{2}\,y_{n-2}(-2) = y_{n-1}$$

$$(1-x^2)\,y_n - \left[2(n-1)x+1\right]y_{n-1} - (n-1)(n-2)\,y_{n-2} = 0$$

Example :- If $y = \sin\log(x^2+2x+1)$, prove that

$$(1+x)^2 y_{n+2} + (2n+1)(x+1)\,y_{n+1} + (n^2+4)\,y_n = 0$$

[2012, 18]

Solution :- Hint $y = \sin\log(x^2+2x+1)$

Diff. w. r. to x

$$y_1 = \cos\log(x^2+2x+1)\left(\frac{1}{x^2+2x+1}\right)(2x+2)$$

$$y_1 = \cos\left[\log(x^2+2x+1)\right]\left(\frac{2}{x+1}\right)$$

$$(x+1)\,y_1 = 2\cos\left[\log(x^2+2x+1)\right]$$

Diff. again w.r. to x

$$y_1 + (x+1)\,y_2 = -2\sin\left[\log(x^2+2x+1)\right]\frac{d}{dx}$$

$$= -2y\cdot\frac{1}{x^2+2x+1}\cdot(2x$$

$$(x+1)\,y_2 + y_1 = -4\,\frac{y}{(x+1)}$$

$$(x+1)^2\cdot y_2 + (x+1)\,y_1 + 4y = 0$$

Now by using Leibnitz theorem (2) n times w. r. to x, we get

$$(1+x)^2 y_{n+2} + (2n+1)(x+1)\,y_{n+1} + (n^2+4)$$

Example :- If $y = e^{a\sin^{-1}x}$ then fi value of $(1-x^2)\,y_2 - x y_1 - a^2 y$.

Solution :- Hint - Diff $y = e^{a\sin^{-1}x}$ 2 times previous examples. ans - 0.

Example :- If $I_n = \frac{d^n}{dx^n}(x^n \log x)$ then show that $I_n = n I_{n-1} + \lfloor n-1$ [2016]

Solution :-

$$I_n = \frac{d^n}{dx^n}(x^n \log x)$$

$$I_n = \frac{d^{n-1}}{dx^{n-1}} \frac{d}{dx}(x^n \log x)$$

$$= \frac{d^{n-1}}{dx^{n-1}}\left[n x^{n-1}\log x + \frac{x^n}{x}\right]$$

$$= n\frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$\boxed{I_n = n I_{n-1} + \lfloor n-1}$$

Lecture 16

To find nth derivative of a function at $x=0$

Example :- If $y = x^2 \exp(2x)$, determine $(y_n)_0$.

Solution :- $y = x^2 e^{2x}$ ... (1)

Diff (1) n times w.r. to x by Leibnitz theorem

$$y_n = 2^n e^{2x} \cdot x^2 + {}^nC_1 \cdot 2^{n-1} e^{2x} \cdot 2x + {}^nC_2\, 2^{n-2} e^{2x} \cdot 2$$

$$y_n = 2^n x^2 e^{2x} + 2^n n x e^{2x} + (n^2 - n) 2^{n-2} e^{2x} \quad \dots(2)$$

by putting $x=0$ in (2) we get

$$(y_n)_{x=0} = (n^2 - n)\, 2^{n-2}$$

V. Imp

Example :- If $y = \sin(a \sin^{-1} x)$, find $(y_n)_0$. [2015, 2018, 2020]

Solution:- $y = \sin(a\sin^{-1}x)$ ...(1)

$$y_1 = \cos(a\sin^{-1}x)\left(\frac{a}{\sqrt{1-x^2}}\right)$$

$$\sqrt{1-x^2}\, y_1 = a\cos(a\sin^{-1}x) \quad \text{...(2)}$$

squaring both sides,

$$(1-x^2)\, y_1^2 = a^2 \cos^2(a\sin^{-1}x)$$

$$= a^2\left(1-\sin^2(a\sin^{-1}x)\right)$$

$$(1-x^2)\, y_1^2 = a^2(1-y^2) \quad \text{...(3)}$$

diff (3) w.r.to x

$$(-2x)\, y_1^2 + 2y_1 y_2(1-x^2) = -2a^2 y y_1$$

$$(1-x^2)\, y_2 - x y_1 + a^2 y = 0 \quad \text{...(4)}$$

diff. (4) n times w.r.to x by using Leibnitz theorem, we get

$$(1-x^2)\, y_{n+2} - (2n+1)x y_{n+1} - (n^2-a^2)\, y_n = 0 \quad \text{...(5)}$$

Now we will find $y_n(0)$,

by putting $x=0$ in equations (1), (2), (3) (4) and (5) we get,

$(y)_0 = 0$

$(y_1)_0 = a$

$(y_2)_0 = 0$

and $(y_{n+2})_0 = (n^2 - a^2)\, y_n(0)$ ...

putting $n = 1, 2, 3, \dots$ in (6) we get

$$y_3(0) = (1^2-a^2)\, y_1(0) = (1^2-a^2)a$$

$$y_4(0) = (2^2-a^2)\, y_2(0) = 0$$

$$y_5(0) = (3^2-a^2)\, y_3(0) = (3^2-a^2)(1^2-a^2$$

$$y_6(0) = (4^2-a^2)\, y_4(0) = 0$$

$\vdots$

In general

$$y_n(0) = \begin{cases} 0 & \text{when } n \text{ is e} \\ [(n-2)^2-a^2][(n-4)^2-a^2]\cdots(3^2-a^2)(1^2-a^2) & \text{when } n \text{ is} \end{cases}$$

Imp Example :- If $y = (\sin^{-1}x)^2$, prove that

$$(y_n)_0 = \begin{cases} 0, & n \text{ is odd} \\ 2.2^2.4^2.6^2\ldots(n-2)^2, & n \text{ is even}, \; n \neq 2 \end{cases}$$ [2013]

Solution :- $y = (\sin^{-1}x)^2$ --- (1)

Diff. both sides w.r.to x

$$y_1 = (2\sin^{-1}x)\left(\frac{1}{\sqrt{1-x^2}}\right) \quad \text{---(2)}$$

squaring both sides

$$(1-x^2)\,y_1^2 = 4(\sin^{-1}x)^2 = 4y$$

Diff. both sides again w.r. to x, we get

$$(-2x)\,y_1^2 + (1-x^2)\,2y_1y_2 = 4y_1$$

$$(1-x^2)\,y_2 - x\,y_1 - 2 = 0 \quad \text{---(3)}$$

Diff both sides n-times w.r.to x by Leibnitz theorem

$$(1-x^2)\,y_{n+2} + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2}\,y_n(-2)$$

$$- x\,y_{n+1} - n\,y_n = 0$$

gives

$$(1-x^2)\,y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0 \quad \text{---(4)}$$

put $x=0$ in (1), (2), (3) and (4) we get

$(y)_0 = 0$ $\qquad (y_2)_0 = 2$

$(y_1)_0 = 0$

$$(y_{n+2})_0 = n^2 (y_n)_0 \quad \cdots (5)$$

Putting $n = 1, 2, 3, ---$ in (5), we get

$$y_3(0) = (y_n)_0 = 1^2 (y_1)_0 = 0$$

$$(y_4)_0 = 2^2 (y_2)_0 = 2^2.2$$

$$(y_5)_0 = 3^2 (y_3)_0 = 0$$

$$(y_6)_0 = 4^2 (y_4)_0 = 4^2.2^2.2$$

$\vdots$

In general

$$y_n(0) = \begin{cases} 0, & n \text{ is odd} \\ (n-2)^2 \cdots 4^2.2^2.2, & n \text{ is even}, \; n \neq 2 \end{cases}$$

Lecture 48 Curve Tracing

The knowledge of curve tracing is to avoid the labour of plotting a large number of points. It is helpful in finding the length of curve, area volume and surface area. The limits of integration can be easily found on tracing the curve.

Steps for curve tracing of algebraic curve

1. Symmetry :- (i) If equation of the curve remains same by replacing y by $-y$, curve is symmetric about x-axis.

   e.g. $y^2 = 4ax$

   (ii) If the curve is same by replacing x by $-x$, symmetry about y-axis

   e.g. $x^2 = 4ay$

   (iii) If on interchanging x and y, curve is same then curve is symmetric about $y=x$ line.

   e.g. $x^3+y^3 = 3axy$

   (iv) If on interchanging (x) and (y) by $(-x)$ and $(-y)$ respectively, curve is same then the curve is symmetric about opposite quadrants

   e.g. $xy = c$

2. (a) Curve through origin :- A curve passes through origin if it does not contain consant term.

   e.g. $y^2 = 4ax$

   (b) Tangent at the origin :- The equation of the tangents at the origin can be obtained by equating to zero the lowest degree term in the equation of the curve.

   e.g. In curve $y^2(a^2+x^2) = x^2(a^2-x^2)$, eq of tangents are $a^2(y^2-x^2)=0$

   $\Rightarrow y = \pm x$

3. The point of intersection with the axes

   (a) By putting $y=0$, we get the point of intersection with x-axis

   e.g. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, put $y=0$, we get $x=\pm$

   (b) By putting $x=0$, we get the pt. of intersection with y-axis

4. Region in which the curve does not lie

   If the curve becomes imaginary for some values of x, the curve does not exist for such e.g. For negative values of x, y is imaginary. In the curve, $y^2 = 4ax$, there is no curve in III & IV quadrants.

5. Asymptotes are tangents to the curve at ∞

(a) Asymptotes parallel to x-axis is obtained by equating to zero, the coefficient of the highest power of x.

e.g. $yx^2 - 4x^2 + x + 2 = 0$, asymtote is $y = 4$

(b) Asymptotes parallel to y-axis is obtained by equating to zero the coefficient of highest power of y.

e.g. $xy^2 = 4a^2(2a - x)$, asymptote parallel to y axis is $x = 0$.

(c) Oblique asymptotes

(i) Find $\phi_n(m)$ by putting $x=1$ and $y=m$ in the highest degree terms of the curve

(ii) solve $\phi_n(m) = 0$ for m.

(iii) Find $\phi_{n-1}(m)$ by putting $x=1$ and $y=m$ in the next highest $(n-1)$ degree terms of the curve

(iv) Find c by $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$

If value of m is repeated two times then find c by $\frac{c^2}{\underline{|2}} \phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$

(V) Obtain the equation of asymptotes by putting the values of m and c in $y = mx + c$.

Example:- Find the oblique asymptote of $xy^2 = 4a^2(2a - x)$, if it has. [2013, 2015]

Solution:- $xy^2 = 4a^2(2a-x)$

$\phi_3(m) = m^2 = 0$ we get $m = 0, 0$

$\phi_2(m) = 0$ and $\phi_3'(m) = 2m$, $\phi_3''(m) = 2$

$\phi_1(m) = 4a^2$

use formula

$$\frac{c^2}{\underline{|2}} \phi_3''(m) + c\,\phi_2'(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2} \cdot 2 + 4a^2 = 0$$

$$c^2 = -4a^2 \quad (c \text{ becomes imaginary})$$

curve has no real asymptote (Oblique).

Note: but the curve has one real asymptote which is parallel to y-axis which is $x = 0$

Example:- Find the asymptotes of $y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2x + 2y + 1 = 0$

Ans:- $y = 2x$, $y = x - 1$, $y = -x - 2$. [2011]

Solution :- We have $y^2 = \dfrac{x^3}{2a-x}$ ...(1)

1. Origin :- Equation does not contain any Constant term. It passes through origin.
2. Tangent at Origin :- $2ay^2 - xy^2 = x^3$

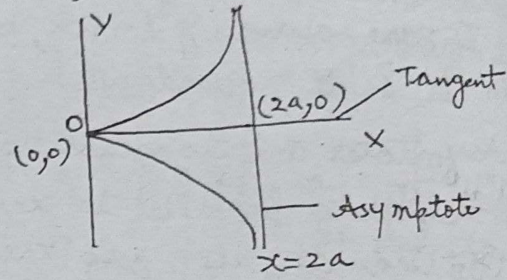
   so put $2ay^2 = 0 \Rightarrow y^2 = 0$

   $y=0$, $y=0$ are two tangents at origin.
3. Symmetry about x-axis :- $\because$ the curve contains only even powers of y, it is symmetric about x-axis.
4. Cusp :- As we have seen, curve has two coincident tangent at origin. Origin is a cusp.
5. Asymptote parallel to y-axis :-

   $$(2a-x)y^2 - x^3 = 0$$

   $2a - x = 0 \Rightarrow x = 2a$ is the asymptote parallel to y-axis.
6. Region of absence of curve :- y^2 becomes negative on putting $x > 2a$ or $x < 0$, therefore, the curve does not exist for $x<0$ and $x>2a$.

So, a sketch of the curve will be



Example :- Trace the curve $y = x(x^2-1)$ [2012]

Solution :- 1. Origin :- Curve passes through the origin

2. Tangent at the origin :- $y + x - x^3 = 0$, by putting lowest degree term to zero

   $y + x = 0 \Rightarrow y = -x$ is the ta

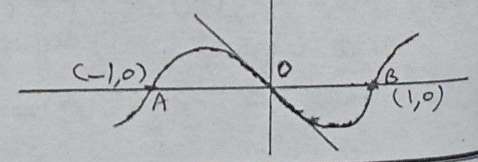
3. Symmetry :- when x is replaced by $-x$ and by $-y$ then equation of the curve remains same, so symmetry in opp quadrants.

4. Point of Intersection with x-axis :- Put $x(x^2-1)$

   $x=0$, $x=1$, $x=-1$

5. Asymptote : There is no asymptote

Sketch of the curve



Example:- Discuss all symmetries of the curve

$$x^2y^2 = x^2 - a^2$$

Solution :- We have $x^2y^2 = x^2 - a^2$

(i) Since the curve has even powers of x, so the curve is symmetric about y-axis.

(ii) Since the curve has even powers of y, so the curve is symmetric about x-axis.

(iii) Since the equation of the curve remains same, if x is replaced by $-x$ and y by $-y$. The curve is symmetric about opposite quadrants.

Steps for Tracing Polar curves:-

1. Symmetrical about the initial line:- On replacing θ by $-\theta$, the equation of the curve remains unchanged, the curve is symmetric about the initial line; for eg:- $r = a(1+\cos\theta)$

2. Symmetrical about pole:- If the powers of r are even, the curve is symmetrical about the origin (pole). for eg:- $r^2 = a^2\cos 2\theta$.

3. Table:- Prepare a table for values of θ and corresponding values of r.

4. Symmetrical about a line ⊥ to initial line and passing through pole:- If on changing θ to $-\theta$ and r to $-r$ simultaneously, the eq^n remains unchanged, the curve is symmetrical about line passing through pole and ⊥ to the initial line; for eg:- $r^2 = a^2\cos 2\theta$.

If on changing θ by $(\pi-\theta)$, the eq^n remains unchanged, the above symmetry exists. $r = a\sin\theta$.

5. Region of the curve:- If for a certain value of θ, r^2 is negative, the curve does not lie there.

6. Greatest or least value of (r):- Find the greatest or least values of r, so the curve lies within or without a certain circle.

7. Tangents at the Pole:- Putting $r=0$, the real values of θ give the tangents at the pole.

for eg:- consider the curve $r=a(1-\cos\theta)$, putting $r=0$, we get $\cos\theta=1 \Rightarrow \theta=0$.

Hence, the curve passes through the pole and the line $\theta=0$ i.e. the initial line is the tangent at the pole.

Again, consider $r=a(3-\sin\theta)$

Putting $r=0$ we get $\sin\theta=3$ which does not give any real value of θ $[\because |\sin\theta|\leq 1]$.

Hence there is no tangent to the curve at pole.

8. Cartesian equation:- In certain cases, it is convenient to convert the given eq^n into cartesian one.

Example :- Trace the curve $r^2 = a^2\cos 2\theta$ [2014, 2...

Solution :- 1. Symmetry:- (i) Since the equation of curve remains unchanged if θ is replaced by $-\theta$. The curve is symmetric about initial line.

(ii) $$r^2 = a^2\cos 2(\pi-\theta) = a^2\cos 2\theta$$

when the angle θ is replaced by $(\pi-\theta)$, the curve remains unchanged. The curve is symmetric about $\theta = \frac{\pi}{2}$

(iii) Since
$$(-r)^2 = a^2\cos 2(\pi+\theta)$$
$$r^2 = a^2\cos 2\theta$$

So, the curve is symmetric about opposite quadrants.

2. Origin or pole :- Put $r=0$ in the eq^n of the curve, we get

$$\cos 2\theta = 0$$
$$2\theta = \pm\frac{\pi}{2} \Rightarrow \theta = \pm\frac{\pi}{4}.$$

The curve passes through the pole and the tangents at the pole are

$$\theta = \pm\frac{\pi}{4}.$$

Also when $\theta=0$, $r=\pm a$

the curve meets the initial line at $(\pm a, 0)$

3. Asymptotes :- The curve has no asymptote.

4. value of ϕ :- $\phi = \frac{\pi}{2} + 2\theta$

when $\theta = 0$, $\phi = \frac{\pi}{2}$

and $r = \pm a$

hence at the points $(a, 0)$ and $(-a, 0)$ the tangents are perpendicular to initial line.

5. Special points and Region

$\because r = a\sqrt{\cos 2\theta}$, $\frac{dr}{d\theta} = \frac{-a \sin 2\theta}{\sqrt{\cos 2\theta}}$

For $0 < \theta < \frac{\pi}{4}$, $\frac{dr}{d\theta}$ is negative

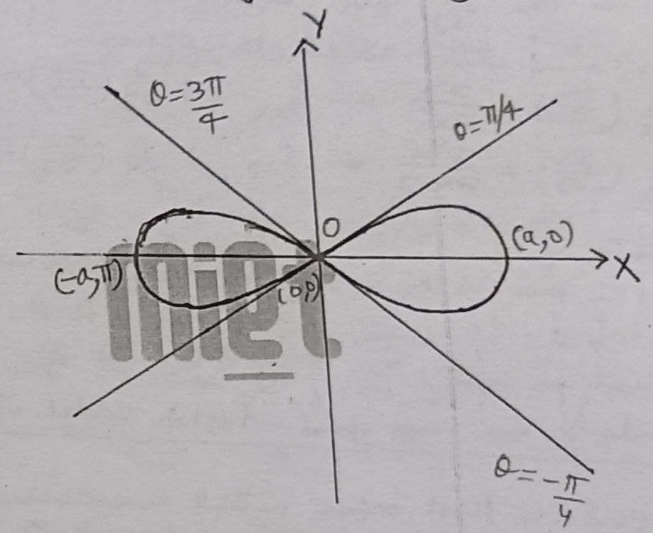
(r decreases in this range)

For $\frac{3\pi}{4} < \theta < \pi$, $\frac{dr}{d\theta}$ is positive

(r increases in this range)

For $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, r imaginary, hence no portion of the curve lies between $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$

Thus, we can trace the part of the curve about the initial line. The part of the curve below the initial line can be traced by symmetry.



Unit- III Differential Calculus - II

Introduction to Partial Differentiation:-

If $z = f(x,y)$ be a function of two independent variables x and y, then to differentiate z partial diff. is used.

Partial derivatives of $z = f(x,y)$

(i) First order partial derivatives are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

* To find $\frac{\partial z}{\partial x}$, diff. z with respect to x, taking y constant
* To find $\frac{\partial z}{\partial y}$, diff z w.r.to y, taking x as constant

(ii) Second order Partial Derivatives –

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ or f_{xx} ; $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$ or f_{yx}

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$ or f_{xy} ; $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$ or f_{yy}

Note (1) $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ or $f_{xy} = f_{yx}$

(2) The process of differentiating $y = f(x)$ is called ordinary diff.

(3) If $u = f(x_1, x_2, \ldots, x_n)$ then first order partial derivatives are $\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}$

Similarly we can find higher order partial derivatives.

Q1. Find the first order partial derivatives of the functions:

(i) $u = y^x$, (ii) $z = \log(x^2+y^2)$, (iii) $z = \cos^{-1}\left(\frac{x}{y}\right)$

Sol. (i) $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(y^x) = y^x \log y$ $\boxed{\text{(Taking } y \text{ as constant then use } \frac{d}{dx} a^x = a^x \log a)}$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(y^x) = x y^{x-1}$ $\boxed{\text{(Taking } x \text{ as constant then use } \frac{d}{dx} x^n = n x^{n-1})}$

(ii) $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\log(x^2+y^2) = \frac{1}{x^2+y^2}\frac{\partial}{\partial x}(x^2+y^2) = \frac{1}{x^2+y^2}(2x+0)$

$= \frac{2x}{x^2+y^2}$ Taking y as constant to find $\frac{\partial z}{\partial x}$

Since $z = \log(x^2+y^2)$ is symmetrical w.r. to x and y (as we get same function after interchanging x and y)

Hence to find $\frac{\partial z}{\partial y}$ interchange x and y.

$\therefore \frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$

(iii) $\frac{\partial z}{\partial x} = -\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{-y}{\sqrt{y^2-x^2}}\cdot\frac{1}{y}(1) = \frac{-1}{\sqrt{y^2-x^2}}$

and

$\frac{\partial z}{\partial y} = -\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{y}{\sqrt{y^2-x^2}}\cdot x\left(-\frac{1}{y^2}\right) = \frac{x}{y\sqrt{y^2-x^2}}$

Q2. If $u(x,y,z) = \log(\tan x + \tan y + \tan z)$, show that

$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$

Sol. Differentiating u partially w.r.t. x, we get

$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z}(\sec^2 x)$

Similarly,

$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z}$ and $\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z}$

L.H.S. $= \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$

$= \frac{\sin 2x \cdot \sec^2 x + \sin 2y \cdot \sec^2 y + \sin 2z \cdot \sec^2 z}{\tan x + \tan y + \tan z}$

$= \frac{2\sin x \cos x \cdot \frac{1}{\cos^2 x} + 2\sin y \cos y \cdot \frac{1}{\cos^2 y} + 2\sin z \cos z \cdot \frac{1}{\cos^2 z}}{\tan x + \tan y + \tan z}$

$= \frac{2\tan x + 2\tan y + 2\tan z}{\tan x + \tan y + \tan z}$

$= 2 =$ R.H.S.

Q.3 If $e^{\frac{-z}{x^2-y^2}} = x-y$ then show that $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = x^2-y^2$ (2016-17)

Sol. Taking log on both sides,

$$\frac{-z}{x^2-y^2} = \log(x-y)$$

$$\Rightarrow -z = (x^2-y^2)\log(x-y)$$

or $z = (y^2-x^2)\log(x-y)$ — ①

Differentiate eqn. ① partially w.r.t. x and y, we get

$$\frac{\partial z}{\partial x} = (0-2x)\log(x-y) + (y^2-x^2)\cdot\frac{1}{x-y}(1-0)$$

$$\therefore\ y\frac{\partial z}{\partial x} = -2xy\log(x-y) + \frac{y(y^2-x^2)}{x-y} \quad \text{— ②}$$

and $$\frac{\partial z}{\partial y} = (2y-0)\log(x-y) + (y^2-x^2)\cdot\frac{1}{x-y}(0-1)$$

$$\therefore\ x\frac{\partial z}{\partial y} = 2xy\log(x-y) - \frac{x(y^2-x^2)}{x-y} \quad \text{— ③}$$

Adding equations ② and ③, we get

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = \frac{(y-x)(y^2-x^2)}{x-y} = -(y^2-x^2) = x^2-y^2$$

Q.4 If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$, find the value of $\frac{\partial^2 u}{\partial x\partial y}$. (2017-18)

Sol. $$\frac{\partial u}{\partial y} = x^2\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} - \left[2y\tan^{-1}\frac{x}{y} + y^2\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot x\left(-\frac{1}{y^2}\right)\right]$$

$$= x\cdot\frac{x^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + x\cdot\frac{y^2}{x^2+y^2}$$

$$= \frac{x^3+xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x(x^2+y^2)}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$\therefore\ \frac{\partial u}{\partial y} = x - 2y\tan^{-1}\frac{x}{y}$$

Hence

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}\frac{x}{y}\right]$$

$$= 1 - 2y\cdot\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2+y^2-2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

Q.5 If $x^2 = au+bv$, $y^2 = au-bv$, prove that

(i) $\left(\frac{\partial u}{\partial x}\right)_y\cdot\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2}$ (2017-18)

(ii) $\left(\frac{\partial v}{\partial y}\right)_x\cdot\left(\frac{\partial y}{\partial v}\right)_u = \frac{1}{2}$

Sol. Given $x^2 = au+bv$ — ① and $y^2 = au-bv$ — ②

Adding ① + ②, $x^2+y^2 = 2au \Rightarrow u = \frac{1}{2a}(x^2+y^2)$ — ③

Subtracting ② from ①, $x^2-y^2 = 2bv \Rightarrow v = \frac{1}{2b}(x^2-y^2)$ — ④

(i) Diff. eq ① partially w.r.t. u,

$2x\left(\frac{\partial x}{\partial u}\right)_v = a\cdot 1 + 0 \Rightarrow \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$

Now diff. eq ③ partially w.r.t. x, $\left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}$

$$\therefore\ \left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a}\cdot\frac{a}{2x} = \frac{1}{2}$$

(ii) Diff. eq ② partially w.r.t. v,

$2y\left(\frac{\partial y}{\partial v}\right)_u = 0 - b\cdot 1 \Rightarrow \left(\frac{\partial y}{\partial v}\right)_u = \frac{-b}{2y}$

Now diff. eq ④ partially w.r.t. y, $\left(\frac{\partial v}{\partial y}\right)_x = \frac{1}{2b}(-2y) = -\frac{y}{b}$

$$\therefore\ \left(\frac{\partial v}{\partial y}\right)_x\cdot\left(\frac{\partial y}{\partial v}\right)_u = \left(\frac{-b}{2y}\right)\left(-\frac{y}{b}\right) = \frac{1}{2}$$

Q. If $u = f(r)$, where $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r) \quad (2015-16)$$

Sol. $r^2 = x^2 + y^2$; diff. partially w.r.t. x & y,

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

Now $u = f(r)$

$$\therefore \frac{\partial u}{\partial x} = f'(r) \cdot \frac{\partial r}{\partial x} = \frac{x}{r} f'(r)$$

Differentiating again w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r} f'(r) + x \cdot \left(-\frac{1}{r^2} \frac{\partial r}{\partial x}\right) f'(r) + \frac{x}{r} f''(r) \cdot \frac{\partial r}{\partial x}$$

$$\left[\because \frac{\partial}{\partial x}(uvw) = vw\frac{\partial u}{\partial x} + uw\frac{\partial v}{\partial x} + uv\frac{\partial w}{\partial x}\right]$$

$$= \frac{1}{r} f'(r) - \frac{x}{r^2} \cdot \frac{x}{r} f'(r) + \frac{x}{r} \cdot \frac{x}{r} f''(r)$$

$$= \left(\frac{1}{r} - \frac{x^2}{r^3}\right) f'(r) + \frac{x^2}{r^2} f''(r)$$

$$= \frac{r^2 - x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r) \quad (\because r^2 = x^2 + y^2)$$

$$= \frac{y^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r)$$

Similarly $\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{r^3} f'(r) + \frac{y^2}{r^2} f''(r)$

($\because$ u is symmetric about x & y, $\therefore$ replace x by y to find $\frac{\partial^2 u}{\partial y^2}$)

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2 + y^2}{r^3} f'(r) + \frac{x^2 + y^2}{r^2} f''(r)$$

$$= \frac{r^2}{r^3} f'(r) + \frac{r^2}{r^2} f''(r) = f''(r) + \frac{1}{r} f'(r)$$

Ans.

Total Derivative

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$, u is called a composite function of (the single variable) t and we can find $\frac{du}{dt}$, which is called the total derivative of u.

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}}$$

Cor. 1. If $z = f(x, y)$, where $x = \phi(u, v)$, $y = \psi(u, v)$, then z is called a composite function of (two variables) u and v and we can find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \end{aligned}}$$

Cor. 2. If $u = f(x, y, z)$ and x, y, z are functions of t then total derivative of u is $\frac{du}{dt}$,

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}}$$

Q1. Find $\frac{du}{dt}$ if $u = x^3 + y^3$, $x = a\cos t$, $y = b\sin t$ (2019-

Sol. $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$$= 3x^2 \cdot (-a\sin t) + 3y^2 (b\cos t)$$

$$= -3(a\cos t)^2 a\sin t + 3(b\sin t)^2 b\cos t$$

$$= 3\sin t\cos t\,[b^3 \sin t - a^3 \cos t]$$

Q2. Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution if $u = x^2+y^2+z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$. (2014-15)

Sol. $\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$

$= 2x\cdot 2e^{2t} + 2y\cdot(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + 2z(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$

$= 4e^{2t}\cdot e^{2t} + 2e^{2t}\cos 3t\,(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + 2e^{2t}\sin 3t\,(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$

$= 4e^{4t} + 4e^{4t}\cos^2 3t - 6e^{4t}\sin 3t\cos 3t + 4e^{4t}\sin^2 3t + 6e^{4t}\sin 3t\cos 3t$

$= 4e^{4t} + 4e^{4t}(\cos^2 3t + \sin^2 3t)$

$= 4e^{4t} + 4e^{4t} = 8e^{4t}$

Verification (by direct substitution)

Put values of x, y and z in given u, we get

$u = (e^{2t})^2 + e^{4t}\cos^2 3t + e^{4t}\sin^2 3t$

$= e^{4t} + e^{4t} = 2e^{4t}$

$\therefore \frac{du}{dt} = 2\cdot 4e^{4t} = 8e^{4t}$.

Q3. If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$ (2020-21)

Sol. Let $X = y-z$, $Y = z-x$, $Z = x-y$

then $u = f(X, Y, Z)$ is a composite function of x, y, z

$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-1) + \frac{\partial u}{\partial Z}(1)$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y} = \frac{\partial u}{\partial X}(1) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-1)$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial X}(-1) + \frac{\partial u}{\partial Y}(1) + \frac{\partial u}{\partial Z}(0)$

Adding, we get

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z} + \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} = 0$

Imp

Q4. If $u = f(2x-3y, 3y-4z, 4z-2x)$, prove that (2019-20)

$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$

Sol. Let $X = 2x-3y$, $Y = 3y-4z$, $Z = 4z-2x$

$\therefore$ u is a composite function of x, y and z.

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x} = \frac{\partial u}{\partial X}(2) + \frac{\partial u}{\partial Y}(0) + \frac{\partial u}{\partial Z}(-2)$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y} = \frac{\partial u}{\partial X}(-3) + \frac{\partial u}{\partial Y}(3) + \frac{\partial u}{\partial Z}(0)$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z} = \frac{\partial u}{\partial X}(0) + \frac{\partial u}{\partial Y}(-4) + \frac{\partial u}{\partial Z}(4)$

L.H.S. $= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$

$= \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} - \frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z} = 0 =$ R.H.S.

Q.5 If $V = f(2x-3y, 3y-4z, 4z-2x)$, prove that (2014-15)

$6\frac{\partial V}{\partial x} + 4\frac{\partial V}{\partial y} + 3\frac{\partial V}{\partial z} = 0$

Sol. Proceed as above Q.

Q6. If $u = f(r, s, t)$ where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$, (2017-18)

show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

Sol. Proceed as above

Hint: Here u is composite function of r, s, t

Cor 3. If $u = f(x,y)$ where $y = \phi(x)$, then u is a composite function of x.

$$\therefore \boxed{\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}}$$

Q.1. If $u = x\log(xy)$, where $x^3+y^3+3xy=1$, find $\frac{du}{dx}$

Sol. $\frac{du}{dx}$ shows that u is function of single variable x, so taking y as a function of x, use the rule $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$ —— ①

Now diff. u, partially w.r.t. x & y respectively,

$$\frac{\partial u}{\partial x} = x\left(\frac{1}{xy}\cdot y\right) + \log xy = 1+\log xy$$

and $\frac{\partial u}{\partial y} = x\cdot\left(\frac{1}{xy}\cdot x\right) = \frac{x}{y}$

Also differentiating $x^3+y^3+3xy=1$, w.r.t. x, we get

$$3x^2+3y^2\frac{dy}{dx}+3\left(x\frac{dy}{dx}+y\right)=0$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{x^2+y}{y^2+x}\right)$$

Put all these values in eq ①,

$$\frac{du}{dx} = 1+\log xy - \frac{x(x^2+y)}{y(y^2+x)} \quad \text{Ans.}$$

Implicit Function- A function $f(x,y)=c$ is called implicit, if it can't be expressible as $x=\phi(y)$ or $y=\phi(x)$.

For Ex. $x^3+y^3+3xy=1$, $x^y+y^x=xy$ etc.

Cor. 4. If $f(x,y)=c$ is an implicit function then

$$\boxed{\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{f_x}{f_y}}$$

Q.2. If $f(x,y)=0$, $\phi(y,z)=0$, show that

$$\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$$

Sol.

$f(x,y)=0$ gives $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

$\phi(y,z)=0$ gives $\frac{dz}{dy} = -\frac{\frac{\partial \phi}{\partial y}}{\frac{\partial \phi}{\partial z}}$

Now $\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx} \Rightarrow \frac{dz}{dx} = \frac{\frac{\partial \phi}{\partial y}\,\frac{\partial f}{\partial x}}{\frac{\partial \phi}{\partial z}\,\frac{\partial f}{\partial y}}$

$$\Rightarrow \frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}$$

Q3. If $f(x,y,z,w)=0$, then find $\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$ (2015-16, 2017-18)

Sol. $\frac{\partial x}{\partial y} = -\frac{f_y}{f_x}$, $\frac{\partial y}{\partial z} = -\frac{f_z}{f_y}$

$\frac{\partial z}{\partial w} = -\frac{f_w}{f_z}$, $\frac{\partial w}{\partial x} = -\frac{f_x}{f_w}$

Now $\frac{\partial x}{\partial y}\cdot\frac{\partial y}{\partial z}\cdot\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x} = \left(-\frac{f_y}{f_x}\right)\left(-\frac{f_z}{f_y}\right)\left(-\frac{f_w}{f_z}\right)\left(-\frac{f_x}{f_w}\right)$

$= 1$.

Homogeneous Functions

A function $f(x,y)$ is said to be homogeneous of degree (or order) n if it can be expressed in the form $x^n \phi\left(\frac{y}{x}\right)$ or $y^n \phi\left(\frac{x}{y}\right)$

An alternative test-

If $f(tx, ty) = t^n f(x,y)$

then $f(x,y)$ is homogeneous function of order n.

Similarly, $f(x,y,z)$ is said to be homogeneous of degree (or order) n if $f(tx, ty, tz) = t^n f(x,y,z)$

Euler's Theorem on Homogeneous Functions.

If u is a homogeneous function of degree n then

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu} \quad \text{where } u \equiv u(x,y)$$

Note ① If u is a homogeneous function of degree n in x, y and z then $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu}$

② If u is a homogeneous function of degree n in x & y then $\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u}$

Q1. If $u = x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$ then find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ (2018-19)

Sol $u(tx, ty) = (tx)^3(ty)^2\sin^{-1}\left(\frac{ty}{tx}\right) = t^5 x^3y^2\sin^{-1}\left(\frac{y}{x}\right)$
$= u(x,y)$

$\therefore$ u is homogeneous function of order $n=5$

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 5u$$

Q2. If $V = (x^2+y^2+z^2)^{-\frac{1}{2}}$ then find $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z}$ (2015-16)

Sol
$V(tx, ty, tz) = [(tx)^2 + (ty)^2 + (tz)^2]^{-\frac{1}{2}}$
$= t^{-1}[x^2+y^2+z^2]^{-\frac{1}{2}} = t^{-1}V(x,y,z)$

$\therefore$ V is homogeneous function of order $n=-1$

By Euler's Theorem,
$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = nV = -V$$

Q3 If $u(x,y) = (\sqrt{x}+\sqrt{y})^5$, find the value of $\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right)$ (2012-13)

Sol $u(tx, ty) = (\sqrt{tx}+\sqrt{ty})^5 = t^{\frac{5}{2}}(\sqrt{x}+\sqrt{y})^5 = t^{\frac{5}{2}}u(x,y)$

$\therefore$ u is homogeneous function of order $n = \frac{5}{2}$.

By Euler's theorem
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u = \frac{5}{2}\left(\frac{5}{2}-1\right)u$$
$$= \frac{5}{2}\cdot\frac{3}{2}u = \frac{15}{4}u.$$

Q4. If $u = x^2yz - 4y^2z^2 + 2xz^3$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$. (2011-12)

Sol. Proceed as above. (Hint: order $n=4$)

Q5. If $z = x^4y^2\sin^{-1}\left(\frac{x}{y}\right) + \log x - \log y$, find $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$

Sol Let $u(x,y) = x^4y^2\sin^{-1}\left(\frac{x}{y}\right)$

and $v(x,y) = \log x - \log y = \log\frac{x}{y}$

Now $u(tx, ty) = t^6x^4y^2\sin^{-1}\left(\frac{x}{y}\right) = t^6 u(x,y)$

$\therefore$ u is a homogeneous function of degree 6.

Also $v(tx, ty) = \log\left(\frac{tx}{ty}\right) = \log\left(\frac{x}{y}\right) = v(x,y)$

$\therefore v$ is a homogeneous function of degree 0.

By Euler's theorem,

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 6u$; and $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = 0$

Adding them, we get

$x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = 6u$

$\Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 6x^4y^2\sin^{-1}\left(\frac{x}{y}\right)$ $(\because z = u+v)$

Imp

Q6. Verify Euler's theorem for the function $z = \frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$

Sol. $z(tx, ty) = \frac{(tx)^{1/3}+(ty)^{1/3}}{(tx)^{1/2}+(ty)^{1/2}} = \frac{t^{\frac{1}{3}}}{t^{\frac{1}{2}}}\frac{(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})} = t^{-1/6}z(x,y)$

$\therefore z(x,y)$ is homogeneous function of order $n = -\frac{1}{6}$

By Euler's Theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -\frac{1}{6}u$ [2015-16, 2017-18]

Verification –

$$\frac{\partial z}{\partial x} = \frac{(x^{1/2}+y^{1/2})(\frac{1}{3}x^{-2/3}) - (x^{1/3}+y^{1/3})(\frac{1}{2}x^{-1/2})}{(x^{1/2}+y^{1/2})^2}$$

$$x\frac{\partial z}{\partial x} = \frac{1}{(x^{1/2}+y^{1/2})^2}\left[\frac{1}{3}(x^{5/6}+x^{\frac{1}{3}}y^{\frac{1}{2}}) - \frac{1}{2}(x^{\frac{5}{6}}+x^{\frac{1}{2}}y^{\frac{1}{3}})\right]$$

$$\frac{\partial z}{\partial y} = \frac{(x^{1/2}+y^{1/2})(\frac{1}{3}y^{-2/3}) - (x^{1/3}+y^{1/3})(\frac{1}{2}y^{-1/2})}{(x^{1/2}+y^{1/2})^2}$$

$$y\frac{\partial z}{\partial y} = \frac{1}{(x^{1/2}+y^{1/2})^2}\left[\frac{1}{3}(x^{1/2}y^{1/3}+y^{5/6}) - \frac{1}{2}(x^{1/3}y^{1/2}+y^{5/6})\right]$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{(-1/6)}{(x^{1/2}+y^{1/2})^2}\left[x^{5/6}+y^{5/6}+x^{\frac{1}{3}}y^{\frac{1}{2}}+x^{\frac{1}{2}}y^{\frac{1}{3}}\right]$$

$$= -\frac{1}{6}\frac{1}{(x^{1/2}+y^{1/2})^2}\left[x^{\frac{1}{2}}(x^{\frac{1}{3}}+y^{\frac{1}{3}}) + y^{\frac{1}{2}}(x^{\frac{1}{3}}+y^{\frac{1}{3}})\right] = -\frac{1}{6}\frac{(x^{\frac{1}{3}}+y^{\frac{1}{3}})}{(x^{1/2}+y^{1/2})} = -\frac{1}{6}u.$$

Hence verified

Deductions from Euler's Theorem

If $F(u) = f(x,y)$, where $f(x,y)$ is a homo function in x and y of degree n, then

(i) $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)}}$

(ii) $\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \phi(u)[\phi'(u)-1]}$

$\boxed{\text{Where } \phi(u) = n\frac{F(u)}{F'(u)}}$

Note- If $F(u) = f(x,y,z)$, where $f(x,y,z)$ is a homogeneous function in x, y and z of degree n

$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = n\frac{F(u)}{F'(u)}}$

Q1. If $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$. Also evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$ (2020-

Sol. Here u is not a homogeneous function. But

$\sec u = \frac{x^3-y^3}{x+y}$ is of the form $F(u) = f(x$

$f(tx, ty) = \frac{(tx)^3-(ty)^3}{tx+ty} = t^2\frac{x^3-y^3}{x+y} = t^2 f(x,y$

$\therefore f(x,y)$ is a homogeneous function of order n

$\therefore$ By Euler's theorem

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)} = 2\frac{\sec u}{\sec u.\tan u} = 2.\cot u$

Now $\phi(u) = 2\cot u$

By deduction from Euler's theorem

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\phi(u)[\phi'(u)-1]$$
$$=2\cot u\,[-2\operatorname{cosec}^2 u-1]$$
$$=-2\cot u\,[2\operatorname{cosec}^2 u+1]$$

Q2. If $u=\sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$, prove that (2017-18)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=2\tan u$$

Sol. u is not a homogeneous function. But

$\sin u=\frac{x^3+y^3+z^3}{ax+by+cz}$, which is of the form $F(u)=f(x,y,z)$

$$f(tx,ty,tz)=\frac{(tx)^3+(ty)^3+(tz)^3}{atx+bty+ctz}=t^2\,\frac{x^3+y^3+z^3}{ax+by+cz}=t^2 f(x,y,z)$$

Hence $f(x,y,z)$ is a homogeneous function of order 2.
By deduction of Euler's theorem.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{F(u)}{F'(u)}=2\frac{\sin u}{\cos u}=2\tan u.$$

Q3. If $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, prove that

(i) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$

(ii) $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\sin u.\cos 3u$

Sol. Here u is not a homogeneous function. But

$F(u)=\tan u=\frac{x^3+y^3}{x-y}=f(x,y)$

is a homogeneous function of order 2.

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{F(u)}{F'(u)}=\frac{2\tan u}{\sec^2 u}=2\frac{\sin u}{\cos u}.\cos^2 u$$
$$=2\sin u\cos u$$
$$=\sin 2u$$

(ii) Here $\phi(u)=\sin 2u$, $\therefore\ \phi'(u)=2\cos 2u$
By deduction from Euler's theorem,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\phi(u)[\phi'(u)-1]$$
$$=\sin 2u\,[2\cos 2u-1]=\sin 4u-\sin 2u$$
$$=2\cos 3u\cdot\sin u.$$

Some Practice Q.

Q4. Prove that $xu_x+yu_y=\frac{5}{2}\tan u$ if $u=\sin^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$ (2014-15)

Q5. Show that $xu_x+yu_y+zu_z=-2\cot u$, where (2013-14)

$$u=\cos^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$$

Imp

Q6. If $u=\sin^{-1}\left(\frac{x^{1/3}+y^{1/3}}{x^{1/2}-y^{1/2}}\right)^{1/2}$, show that (2011-12)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\tan u.$$

Q7. If $u=\sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right)$, then evaluate the value of (2016-17)

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

Ans $\frac{1}{144}\tan u\,(\sec^2 u-12)$

Imp

Q8. If $u=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that (2018-19)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0.$$

Q9. If $u=\cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then find the value of (2019-20)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

(Ans $-\frac{1}{2}\cot u$)

5 Year's

University Paper Questions

(AKTU Question Bank)

10 Years AKTU University Examination Questions		Unit-2	
S. No	Questions	Session	Lecture No
1	If $y = \sin nx + \cos nx$ prove that $y_r = n^r [1 + (-1)^r \sin 2nx]^{1/2}$ where y_r is the r^{th} differential coefficient of y with respect to x.	2011-12 2017-18	9
2	Find y_n if $y = \frac{x^n - 1}{x - 1}$	2011-12	9
3	Find the n^{th} derivative of $x^{n-1} \log x$.	2011-12 2017-18	9
4	Find the n^{th} derivative of $y = x^2 \sin x$.	2013-14	10
5	If $y = \left(\frac{1+x}{1-x}\right)^{1/2}$, prove that $(1-x^2)y_n - [2(n-1)x+1]y_{n-1} - (n-1)(n-2)y_{n-2} = 0$	2011-12	11
6	If $y = \sin\log(x^2+2x+1)$, Prove that $(1+x)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$.	2012-13 2018-19	11
7	If $y = e^{\tan^{-1}x}$, prove that $(1+x^2)\, y_{n+2} + [(2n+2)x - 1]\, y_{n+1} + n(n+1)\, y_n = 0$	2013-14 2017-18 2020-21	11
8	If $y^{1/m} + y^{-1/m} = 2x$, Prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$	2014-15	11
9	If $y = e^{m\cos^{-1}x}$ then find the relation between y_n, y_{n+1} and y_{n+2}.	2015-16 2019-20	11
10	If $y = e^{a\sin^{-1}x}$ then find $(1-x^2)\, y_2 - x\, y_1 - a^2 y$.	2015-16	10
11	If $I_n = \frac{d^n}{dx^n}(x^n \log x)$ then show that $I_n = n\, I_{n-1} + (n-1)!$	2016-17	10
12	If $y = x^2 \exp(2x)$ determine $(y_n)_0$.	2012-13	11
13	If $y = (\sin^{-1} x)^2$ or $\sin\sqrt{y} = x$ prove that $y_n(0) = 0$ for n odd and $y_n(0) = 2.2^2.4^2.6^2 .... (n-2)^2$, $n \neq 2$ for n even	2013-14	11
14	If $y = \sin(a\sin^{-1}x)$, find $(y_n)_0$	2015-16 2018-19 2020-21	11
15	Trace the curve $y^2(2a - x) = x^3$.	2011-12 2014-15	40
16	Trace the curve $y = x(x^2 - 1)$.	2012-13	40

17	Trace the curve $r^2 = a^2 \cos 2\theta$.	2014-15 2019-20	40
18	If $f(x, y, z, w) = 0$, then find $\frac{\partial x}{\partial y} \times \frac{\partial y}{\partial z} \times \frac{\partial z}{\partial w} \times \frac{\partial w}{\partial x}$.(Very Short question)	2015-16	13
19	If $x^2 = au + bv, y^2 = au - bv$ Evaluate $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v$. (Very Short)	2017-18	12
20	If $w = \sqrt{x^2 + y^2 + z^2}$ & $x = \cos v, y = u \sin v, z = uv$, then prove that $\left[u\frac{\partial w}{\partial u} - v\frac{\partial w}{\partial v}\right] = \frac{u}{\sqrt{1+v^2}}$. (Long question)	2016-17	15
21	Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution if $u = x^2 + y^2 + z^2$ and $x = e^{2t}, y = e^{2t} \cos 3t, z = e^{2t} \sin 3t$.(Short question)	2014-15	14
22	If $V = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $6\frac{\partial V}{\partial x} + 4\frac{\partial V}{\partial y} + 3\frac{\partial V}{\partial z} = 0$. (Short question)	2014-15	14
23	If $u = f(r)$ where $r^2 = x^2 + y^2$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$. (Long question)	2015-16	14
24	If $u = f(r, s, t)$, where $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.(Short question)	2017-18	14
25	If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$.(Long question)	2019-20	14
26	Find $\frac{du}{dt}$ if $u = x^3 + y^3, x = a \cos t, y = b \sin t$.(Very Short)	2019-20	12
27	If $u = f(y - z, z - x, x - y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.(Very Short)	2020-21	14
28	If $u = x^2yz - 4y^2z^2 + 2xz^3$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.(Very Short question)	2011-12	15
29	If $u(x, y) = (\sqrt{x} + \sqrt{y})^5$, find the value of $\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right)$. (Very Short question)	2012-13	16
30	Show that: $xU_x + yU_y + zU_z = -2 \cot u$, where $u = \cos^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$ (Short question)	2013-14	16
31	Prove that $xu_x + yu_y = \frac{5}{2} \tan u$ if $u = \sin^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$.(Short question)	2014-15	15
32	Verify Euler's theorem for the function $z = \frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$. (Long question)	2015-16	15

33	If $V = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, then find $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z}$.(Very Short question)	2015-16	14
34	If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right); xy \neq 0$ prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$. (Long question)	2017-18	14
35	If $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2 \tan u$.(Long question)	2017-18	16
36	If $u = x^3y^2 \sin^{-1}\left(\frac{y}{x}\right)$, then find then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. (Very Short)	2018-19	15
37	If $u = \sin^{-1}\left(\frac{x^{\frac{1}{3}}+y^{\frac{1}{3}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}\right)^{\frac{1}{2}}$. Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{12} \tan u$.(Short question)	2011-12	15
38	If $u = \sin^{-1}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right)$, then evaluate the value of $\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right)$. (Short question)	2016-17	16
39	If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2} \cot u = 0$. (Long question)	2018-19	15
40	If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$. (Very Short)	2019-20	15
41	If $u = \sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2 \cot u$. Also evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. (Long question)	2020-21	16